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Abstract. Spin models on quenched random graphs are related to many important optimization problems. We give a new derivation of their mean-field equations that elucidates the role of the natural order parameter in these models.

Spin models on quenched random graphs have been studied extensively in recent years [1–5] for a couple of reasons. First, a large class of difficult (and interesting in practice) optimization problems such as graph partitioning and graph colouring [6] can be formulated as a search for the ground state of such models. Their zero-temperature limit could thus yield valuable information on average properties of the optimal solutions. Second, for finite connectivity, such models are closer to realistic systems than their infinite-range counterparts, yet mean-field theory is expected to stay exact. They thus provide a simpler setting in which to try to test whether the ultrametric structure and other properties of Parisi's solution of the spin-glass phase [7] survive for finite-range interactions.

In this paper, we would like to give a new derivation of the mean-field equations [4, 5] for such models. It is based on some simple arguments, well known from the study of matrix models of 2D gravity [8, 9] and of the large-order behaviour of perturbative series [10], and adapted here in the context of disordered systems. Besides being simple and exact, this novel derivation elucidates the role of the natural order parameter in these models [4, 5]. Furthermore, it can be adapted readily to a variety of different situations. We will not address here the difficult problem of solving these equations in the spin-glass phase. We will, however, comment briefly on the phase diagram in the case of pure ferromagnetic or antiferromagnetic couplings, as well as on the interpretation of the underlying graphs as infinite-genus triangulations.

Consider first the ensemble of all trivalent (ϕ^3) graphs made out of $2n$ vertices. If one ignores accidental-symmetry factors, the number of such graphs is given by the integral expression

$$\mathcal{N}_n = \frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}\lambda}{\lambda^{2n+1}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}\phi}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\phi^2+\frac{1}{6}\lambda\phi^3}. \tag{1}$$

Indeed, the ϕ-integral can be expressed as a sum over all topologically-distinct ϕ^3 graphs weighted with $\lambda^{\#\ \text{vertices}}$ multiplied by an inverse symmetry factor. The contour λ-integral then picks out only the contribution of graphs with precisely $2n$ vertices. In the large-n limit, we can evaluate this integral at the dominant non-trivial saddle points for both variables

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ϕ and λ. After a rescaling of variables ($\phi \to \phi/\lambda$) and some straightforward Gaussian integrations, the result of the calculation reads

$$\mathcal{N}_n = (n/e)^n \hat{S}^{-n} (-2\pi n \det \hat{S}'')^{-1/2} (1 + \mathrm{O}(1/n)). \tag{2}$$

Here $S = \frac{\phi^2}{2} - \frac{\phi^3}{6}$ is the rescaled 'action' of the theory, $\hat{S}$ its value at the dominant non-trivial saddle point $\hat{\phi}$ which solves the 'field equation'

$$\frac{\partial S}{\partial \phi} = 0 \tag{3}$$

and $\det \hat{S}''$ is simply the second derivative of S at the saddle point. Equation (2) is a standard result for the large-order behaviour of perturbative expansions [10] and will stay valid in the more complicated cases studied below. In the case at hand, using $\hat{\phi} = 2$, $\hat{S} = \frac{2}{3}$ and $-\hat{S}'' = 1$, one recovers the correct counting of large undecorated ϕ^3 graphs, whose precise number $\mathcal{N}_n$ is

$$\mathcal{N}_n = \left(\frac{1}{6}\right)^{2n} \frac{(6n-1)!!}{(2n)!}.$$

Let us consider now an Ising model with spins $\sigma_i = \pm 1$ lying on the $2n$ vertices of a ϕ^3 graph $\mathcal{G}_n$. The partition function is

$$Z_{\mathcal{G}_n}(J, h) = \sum_{\sigma_1, \ldots, \sigma_{2n}} \exp\left(J \sum_{\langle ij \rangle} \sigma_i \sigma_j + h \sum_i \sigma_i \right) \tag{4}$$

where the sums in the Boltzmann weight run over all edges and vertices, respectively, of the graph $\mathcal{G}_n$, J is the spin–spin coupling and h is a magnetic field. The average of the partition function over all graphs can be expressed as an integral [9] over a 'field' defined on the discrete space $\{+, -\}$

$$\overline{Z_{\mathcal{G}_n}(J, h)} \times \mathcal{N}_n = \frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}\lambda}{\lambda^{2n+1}} \int \frac{\mathrm{d}\phi_+ \, \mathrm{d}\phi_-}{2\pi \sqrt{\det \Delta}} \exp(-S) \tag{5}$$

where

$$S = \tfrac{1}{2} \sum_{\sigma, \tau} \phi_\sigma (\Delta^{-1})_{\sigma\tau} \phi_\tau - \tfrac{1}{6} \lambda (\mathrm{e}^h \phi_+^3 + \mathrm{e}^{-h} \phi_-^3) \tag{6}$$

and the 2×2 'propagator' matrix has entries

$$\Delta_{\sigma\tau} = \mathrm{e}^{J\sigma\tau}. \tag{7}$$

Indeed, the weak-λ expansion of the ϕ_σ integral(s) is given as before by the sum over ϕ^3 Feynman diagrams, while the λ-integration forces the number of vertices to be $2n$. For any given diagram, the vertices are now, however, labelled by a 'position in real space' $\sigma_i = \pm$. Furthermore, there is a weight $\mathrm{e}^{h\sigma_i}$ for each vertex and a propagator $\mathrm{e}^{J\sigma_i\sigma_j}$ for each edge. Summing over all 'positions' of vertices thus yields the partition function of the Ising model on the corresponding graph. This justifies equation (5). Note that the ϕ_σ integral is, strictly speaking, only defined through its asymptotic expansion.

In the thermodynamic limit of large graphs ($n \to \infty$), we can again calculate the above integral by the saddle-point technique. We limit ourselves, for simplicity, to the case of vanishing magnetic field. The action (6) has three non-zero saddle points, which after the usual rescaling read

$$\hat{\phi}_+ = \hat{\phi}_- = \frac{2\sqrt{g}}{g+1} \tag{8}$$

and

$$\hat{\phi}_\pm = \frac{\sqrt{g}}{(g-1)}\left(1 \pm \sqrt{\frac{g-3}{g+1}}\right) \qquad \text{or} \qquad \hat{\phi}_+ \leftrightarrow \hat{\phi}_-. \tag{9}$$

Here, $g \equiv \mathrm{e}^{2J}$, so that $g \in [1, \infty)$ corresponds to ferromagnetic couplings $J > 0$, while $g \in [0, 1]$ corresponds to antiferromagnetic couplings $J < 0$. The (degenerate) saddle points (9) dominate in the low-temperature ferromagnetic region $g > 3$ but become complex below $g = 3$ where the saddle point (8) takes over. This can be continued analytically down to $g = 0$, i.e. to the zero-temperature antiferromagnet. The transition at $g = 3$ corresponds, in fact, to the onset of ferromagnetic order. This can be seen from the expression for the average (annealed) magnetization

$$\mathcal{M}_{\mathrm{ann}} \equiv \frac{1}{2n}\frac{\partial}{\partial h}\log\overline{Z_{\mathcal{G}_n}}\bigg|_{h=0} = \frac{\hat{\phi}_+^3 - \hat{\phi}_-^3}{\hat{\phi}_+^3 + \hat{\phi}_-^3} = \begin{cases} \pm\dfrac{g}{g-2}\sqrt{\dfrac{g-3}{g+1}} & \text{if } g > 3 \\ 0 & \text{if } g < 3 \end{cases} \tag{10}$$

which follows by straightforward manipulations. For completeness, we also give the result for the average partition function, valid up to terms of order $\mathrm{O}(1/n)$

$$\begin{aligned} \log\overline{Z_{\mathcal{G}_n}(g)}\bigg|_{h=0} &= -n\log\tfrac{3}{2}\hat{S} - \tfrac{1}{2}\log(-\det(\Delta\hat{S}'')) \qquad\qquad (11) \\ &= \begin{cases} -n\log\dfrac{2g\sqrt{g}}{(g+1)^3} - \dfrac{1}{2}\log\dfrac{3-g}{g+1} & \text{if } g < 3 \\ -n\log\dfrac{3\sqrt{3}}{32} + \dfrac{1}{4}\log n + \log\left(\Gamma\left(\dfrac{1}{4}\right)\left(\dfrac{3}{4}\right)^{1/4}\Big/\sqrt{\pi}\right) & \text{if } g = 3 \\ -n\log\dfrac{g(g-2)\sqrt{g}}{(g-1)^3(g+1)} - \dfrac{1}{2}\log\dfrac{g-3}{g-1} & \text{if } g > 3. \end{cases} \end{aligned}$$

Note that the logarithmic corrections at the critical point are due to the appearance of a zero mode, so that in the calculation of the integral, we must keep terms higher than quadratic in the action. These logarithmic corrections are a manifestation of the long-range order. Note also that in the ferromagnetic region, we took into account only one of the two saddle points, corresponding to a pure thermodynamic state.

Up to now, we have treated the random graphs as annealed disorder, meaning that they are allowed to participate in the dynamics on an equal footing with the Ising spins. We can quench them by employing the replica trick

$$\overline{\log Z} = \lim_{k\to 0}\frac{\overline{Z^k} - 1}{k}. \tag{12}$$

To this effect, we introduce a real field with argument on the hypercube in k dimensions $\phi(\{\sigma\}) \equiv \phi(\sigma^1, \ldots, \sigma^k)$. Each vertex of a Feynman diagram will now be labelled by the values of k distinct spins, one for each replica†. Arguing as before, we can express the kth moment of the Ising partition function in zero magnetic field as follows

$$\overline{Z^k_{\mathcal{G}_n}} \times \mathcal{N}_n = \frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}\lambda}{\lambda^{2n+1}} \frac{1}{\sqrt{\det \Delta}} \int \prod_{\{\sigma\}} \frac{\mathrm{d}\phi(\{\sigma\})}{\sqrt{2\pi}} \exp(-S) \tag{13}$$

with

$$S = \tfrac{1}{2} \sum_{\{\sigma\}\{\tau\}} \phi(\{\sigma\}) \Delta^{-1}(\{\sigma\}, \{\tau\}) \phi(\{\tau\}) - \tfrac{1}{6}\lambda \sum_{\{\sigma\}} \phi(\{\sigma\})^3. \tag{14}$$

Here, $\sum_{\{\sigma\}}$ stands for a sum over all possible values of the k spins σ^a and the $2^k \times 2^k$ propagator matrix has entries corresponding to the Boltzmann weight of k non-interacting replicas on an edge $\Delta(\{\sigma\}, \{\tau\}) = \exp\left(J \sum_a \sigma^a \tau^a\right)$. More generally, we may allow a propagator

$$\Delta(\{\sigma\}, \{\tau\}) = \int \mathrm{d}J \, \rho(J) \exp\left(J \sum_a \sigma^a \tau^a\right) \tag{15}$$

which amounts to choosing uncorrelated couplings on each edge with some (arbitrary) distribution $\rho(J)$. We may also trade the $\frac{1}{6}\lambda\phi^3$ interaction for a more general monomial $\frac{\lambda^{M-2}}{M!}\phi^M$ so as to obtain graphs with fixed connectivity equal to M. Extremizing the (rescaled) action finally yields the saddle-point equations

$$\phi(\{\sigma\}) = \frac{1}{(M-1)!} \sum_{\{\tau\}} \Delta(\{\sigma\}, \{\tau\}) \phi(\{\tau\})^{M-1}. \tag{16}$$

The calculation of the partition-function integer moments is thus reduced in the thermodynamic limit to a finite algebraic problem.

In order to quench the random graphs, we, of course, still have to continue, analytically, to values of k near zero. To do this one must make an ansatz on the precise pattern of replica-symmetry breaking. Full symmetry for instance would imply that the field only depends on the fraction of replicas pointing up: $\phi(\{\sigma\}) = \phi(\sigma^1 + \cdots + \sigma^k)$. A first stage of hierarchical breaking, on the other hand, would correspond to the ansatz: $\phi(\{\sigma\}) = \phi(\sigma^1 + \cdots + \sigma^{\frac{k}{m}}, \ldots, \sigma^{k-\frac{k}{m}+1} + \cdots + \sigma^k)$. Details on the $k \to 0$ continuation as well as on the resulting free energy can be found in [4, 5]. Here, we would only like to point out that the order parameter $\phi(\{\sigma\})$ can be related to the more standard magnetization overlaps by the same kind of argument that leads us to equation (10). Indeed, the fraction of vertices with a given value $\{\sigma\}$ for the spins of the k replicas can be easily seen to be proportional to $\hat{\phi}(\{\sigma\})^M$. The definition of the magnetization overlaps, on the other hand, is

$$Q^{a_1 \ldots a_\ell} \equiv \lim_{n\to\infty} \frac{1}{2n} \sum_i \overline{\langle \sigma_i^{a_1} \rangle \cdots \langle \sigma_i^{a_\ell} \rangle} \tag{17}$$

† Note that upper indices label the replicas. They should not be confused with lower indices which label the $2n$ vertices of a graph.

where $\langle A\rangle$ denotes the thermal average, while $\overline{A}$ stands for the average over the quenched disorder. It follows by straightforward manipulations that

$$Q^{a_1\ldots a_\ell}=\frac{\sum_{\{\sigma\}}\sigma^{a_1}\ldots\sigma^{a_\ell}\hat{\phi}(\{\sigma\})^M}{\sum_{\{\sigma\}}\hat{\phi}(\{\sigma\})^M}. \tag{18}$$

Our derivation of equations (16) and (18) is the main result of this paper. Equations (16) have been derived previously for spin models on the Bethe lattice [11] and were later argued to hold for random graphs [4] because such graphs have a tree-like local structure. In [4, 11], the relation of $\hat{\phi}(\{\sigma\})$ to the overlaps differs, however, from equation (18). It is conceivable that this difference can be traced to the effect of finite loops that are ignored in these references. In any case, besides being exact, our novel derivation elucidates the role of the natural order parameter $\phi(\{\sigma\})$ in such models. As we have shown, it is the field generating the diagrammatic expansion and whose mean-field equations yield the instanton that governs the behaviour of this expansion at large orders.

The above analysis can be extended easily to several different contexts. Together with the constraint $Q^a=0\ \forall\, a$, equations (16) are, for instance, the mean-field equations for the graph bipartitioning problem [3, 6]. Fluctuating connectivity can also be accommodated if we trade the monomial interaction with a more general potential $V(\lambda\phi)/\lambda^2$. The saddle-point equations then read

$$\phi(\{\sigma\})=\sum_{\{\tau\}}\Delta(\{\sigma\},\{\tau\})V'(\phi(\{\tau\})) \tag{19}$$

where V' denotes the derivative of V. Note that the λ-integration now fixes the difference of the numbers of edges and vertices. Other constraints can be imposed by extra contour integrations. Equations (19) with an exponential potential $V=\mathrm{e}^{\alpha(\phi-1)}/\alpha$ have also been obtained by De Dominicis and Mottishaw [5] in the case of an ensemble of graphs where the connectivity is a random variable with Poissonian distribution of average α. Finally, as it should be evident, Potts or continuous spins can be introduced by letting the argument of the field ϕ live on the corresponding space.

In the special case of fixed ferromagnetic or antiferromagnetic coupling J, the mean-field equations (16) with $M=3$ admit an obvious set of (factorized) solutions

$$\hat{\phi}(\{\sigma\})=2^{1-k}\hat{\phi}_{\sigma_1}\ldots\hat{\phi}_{\sigma_k} \tag{20}$$

where each factor on the right-hand side stands for (any) solution of the $k=1$ (annealed) problem. When the saddle point (20) dominates, both the overlaps and the leading exponential piece of $\overline{Z^k_{\mathcal{G}_n}}$ factorize, so that despite the average over graphs, the replicas are completely decorrelated†. Continuing $k\to 0$, one finds a quenched free energy equal to the annealed energy (11) up to finite-size corrections‡. The corresponding entropy per spin is

$$\overline{s}=\begin{cases}\dfrac{1}{2}\log\dfrac{(g+1)^3}{2}-\dfrac{3g}{2(g+1)}\log g & \text{if } g<3\\[2ex] \dfrac{1}{2}\log\dfrac{(g-1)^3(g+1)}{g-2}-\dfrac{3g(g^2-2g-1)}{2(g-2)(g-1)(g+1)}\log g & \text{if } g>3.\end{cases} \tag{21}$$

† Decorrelated groups of replicas would correspond more generally to a product solution $\hat{\phi}(\{\sigma\})=2^{1-m}\hat{\phi}_{(k_1)}\ldots\hat{\phi}_{(k_m)}$ where $\hat{\phi}_{(k_\nu)}$ is any solution of the saddle-point equations with k_ν replicas and $\sum_{\nu=1}^{m}k_\nu=k$. Such solutions break the symmetry of replicas and are never dominant for integer k.

‡ It can be verified more generally under the assumption of replica symmetry that the factorized solution (20) is indeed dominant in the $k\to 0$ limit.

It becomes negative below $g \simeq 0.211$, signalling the existence of a phase transition in the low-temperature antiferromagnetic region. This is also confirmed by an analysis of the moments $\overline{Z^k_{\mathcal{G}_n}}$ of the partition function. By completely solving equations (16) ($M = 3$) for $k = 2$, 3 and 4, we have found transition points $g_c^{(2)} \simeq 0.172$, $g_c^{(3)} \simeq 0.187$ and $g_c^{(4)} \simeq 0.205$, below which the factorizable saddle point (20) ceases to dominate, so that $\lim_{n\to\infty} \frac{1}{2n} \log \overline{Z^k_{\mathcal{G}_n}} \neq \lim_{n\to\infty} \frac{k}{2n} \log \overline{Z_{\mathcal{G}_n}}$. This situation is reminiscent of the random-energy model [12], except that the critical temperatures seem to accumulate to a finite value ($g < 1$). The nature of this low-temperature phase deserves some further study. Indeed, although the couplings are purely antiferromagnetic, there is both frustration and disorder since the random graph has loops of arbitrary size.

We conclude with some comments on the interpretation of random graphs as infinite-genus triangulations. This comes about by considering the real field ϕ as an $N \times N$ Hermitian matrix with $N = 1$, so that our ensemble consists of 'fat' graphs $\mathcal{G}_n$ or dual triangulations $\mathcal{G}_n^*$ [8] weighted equally for all genera. The average Euler characteristic can be computed easily by taking a derivative with respect to the size N of the Hermitian matrix with the result

$$\overline{\chi} = -n + \log 6n - \left.\frac{\partial \log \Gamma(x)}{\partial \log x}\right|_{x=1}. \tag{22}$$

Note that since for vacuum ϕ^3 graphs with $2n$ vertices, $\chi = -n + \#$ faces; an average graph in this ensemble has a maximal density of handles. Though rather singular, this 2D surface interpretation allows a mapping of the Ising model on $\mathcal{G}_n$, onto a model with spins lying on the vertices of the dual triangular net $\mathcal{G}_n^*$. This duality is implemented by a linear transformation of the fields that diagonalizes the quadratic part of the action. For $k = 1$, for instance, the action would take the form

$$S = \frac{1}{2}(\tilde{\phi}_+^2 + \tilde{\phi}_-^2) - \frac{\tilde{\lambda}}{2}\left(\frac{\tilde{g}}{3}\tilde{\phi}_+^3 - \tilde{\phi}_+\tilde{\phi}_-^2\right) \tag{23}$$

with

$$\tilde{g} = \frac{g+1}{g-1}. \tag{24}$$

Since the propagator is now diagonal, we can assign a sign $\pm$ to each edge of the $\tilde{\phi}^3$ graph, or equivalently to the dual edge $\langle ij \rangle$ on the triangular lattice $\mathcal{G}_n^*$. We interpret this sign as the value of $\sigma_i\sigma_j$ where the σ's now stand for the spins residing on the vertices of the triangular lattice. The product of three signs around a triangle should be $+$, consistent with the fact that only two kinds of vertices survive in the action (23). Furthermore, there is an extra weight $\tilde{g}$ when all three spins around the triangle are aligned. As can be easily verified, the duality transformation (24) maps the high- and low-temperature ferromagnetic regions of the Ising models on $\mathcal{G}_n$ and $\mathcal{G}_n^*$ to one another. The fact that mean-field theory is exact can be understood in the dual language as a consequence of the fact that the number of vertices grows only logarithmically with n while the connectivity is extensive. Note, finally, that the antiferromagnetic region on $\mathcal{G}_n^*$ corresponds to $\tilde{g} \in [0, 1)$ and is mapped onto the interval $(-\infty, -1]$. The analysis of the moments and entropy shows no signal for a phase transition in this region.

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